BMO Spaces

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Definition 1. *Let $f \in L^1_{loc}(\mathbb{R}^n)$, we call f is in $BMO(\mathbb{R}^n)$ if*

$$\|f\|_{BMO} \coloneqq \sup_{\substack{Q \subset \mathbb{R}^n \\ Q \text{ is a cude}}} \fint_Q \left| f - \fint_Q f \right| < +\infty$$

It is worth noting that the definitions are equivalent if we replace cubes Q with balls B.

We will use f_Q to denote the average of the integral $\fint_Q f$.

Theorem 1. *$\| \cdot \|_{BMO}$ is equivalent to $\| \cdot \|_{BMO,p}$ which is defined as*

$$\|f\|_{BMO,p} \coloneqq \sup_{\substack{Q \subset \mathbb{R}^n \\ Q \text{ is a cude}}} \left(\fint_Q |f - f_Q|^p \right)^{1/p}, \quad 1 \le p < \infty$$

Proof. To prove the two norms are equivalent, we only need to prove $\|f\|_{BMO} \le C\|f\|_{BMO,p}$ and $\|f\|_{BMO,p} \le C\|f\|_{BMO}$.

For the one side: we apply the Hölder's inequality:

$$\fint_Q |f - f_Q| \le \frac{1}{|Q|} \left(\int_Q |f - f_Q|^q \, dx \right)^{1/p} \cdot |Q|^{1/q} = \left(\fint_Q |f - f_Q|^q \, dx \right)^{1/p}$$

Take sup on the both side will give $\|f\|_{BMO} \le \|f\|_{BMO,p}$.

For the other side, we need to apply the John–Nirenberg lemma:

Lemma 1. *If f is in $BMO(\mathbb{R}^n)$, then for $\forall \lambda > 0$, we have*

$$|\{x \in Q : |f - f_Q| > \lambda\}| \le C_1 \exp\left\{ -C_2 \frac{\lambda}{\|f\|_{BMO}} \right\} \cdot |Q|$$

where $\exp(x) = e^x$, C_1 and C_2 are two positive constants.

We will prove the John–Nirenberg lemma in the last part of this note, now let's see firstly how to apply this lemma to prove our theorem 1.

Assume $\|f\|_{BMO} < +\infty$, we have

$$\begin{aligned}
\int_Q |f - f_Q|^p &= p \int_0^\infty \lambda^{p-1} \left|\{x \in Q : |f - f_Q| > \lambda\}\right| \, d\lambda \\
&\le C|Q| \int_0^\infty \lambda^{p-1} \exp\left\{ -C_2 \cdot \frac{\lambda}{\|f\|_{BMO}} \right\} \, d\lambda \\
(\lambda = \tau \|f\|_{BMO}) &\le C|Q| \cdot \|f\|_{BMO}^p \int_0^\infty \tau^{p-1} e^{-C_2 \lambda} \, d\tau \\
&= C(p, n) \cdot |Q| \cdot \|f\|_{BMO}^p
\end{aligned}$$

Thus we have

$$\left(⨍_Q |f - f_Q| \right)^{1/p} \le C \cdot \|f\|_{BMO}$$

Take sup of Q, we have $\|f\|_{BMO,p} \le C \cdot \|f\|_{BMO}$. □

The rest of the note will devote to the proof of John–Nirenberg Lemma.

Proof. We will prove that, there exists a $\alpha(n) > 0$ such that

$$\frac{1}{|Q|} \int_Q \exp\left\{ \alpha \cdot \frac{|f - f_Q|}{\|f\|_{BMO}} \right\} \le C$$

for $\forall f \in BMO(\mathbb{R}^n)$. The the conclusion of John-Nirenberg Lemma follows by Chebyshev's inequality:

$$\left| \{x \in Q : |f - f_Q| > \lambda\} \right| \le \int_Q \exp\left\{ \alpha \cdot \frac{|f - f_Q|}{\|f\|_{BMO}} \right\} \cdot \exp\left\{ -\alpha \frac{\lambda}{\|f\|_{BMO}} \right\}$$

To prove the integral inequality, without loss of generality, we could assume f is smooth and $\|f\|_{BMO} = 1$. For $\mu > \|f\|_{BMO}$, by the Caldrón-Zygmund decomposition, we decompose Q into sub-cudes Q_i and $Q \backslash \cup Q_i$ such that

1. In each Q_i, $\mu < ⨍_{Q_i} |f - f_Q| < 2^n \mu$
2. In $Q \backslash \cup Q_i$, $|f - f_Q| \le \mu$

Now we evaluate the integral of $e^{\alpha|f - f_Q|}$ on Q_i and $Q \backslash \cup Q_i$, we have

$$\int_{Q \backslash \cup Q_i} e^{\alpha|f - f_Q|} \le |Q \backslash \cup Q_i| \cdot e^{\alpha\mu}$$

$$\int_{Q_i} e^{\alpha|f - f_Q|} \le \int_{Q_i} e^{\alpha|f - f_{Q_i}|} \cdot e^{\alpha|f_{Q_i} - f_Q|}$$

Notice $|f_{Q_i} - f_Q| = \left| ⨍_{Q_i} f - f_Q \right| \leq 2^n \mu$ Thus we have

$$\int_Q e^{\alpha|f-f_Q|} \leq |Q \backslash \cup Q_i| e^{\alpha\mu} + \sum_{Q_i} e^{2^n \alpha\mu} \int_{Q_i} e^{\alpha|f-f_{Q_i}|}$$

We define β as follows, our goal is to prove $\beta < +\infty$ for any function f with $\|f\|_{BMO} = 1$.

$$\beta := \sup_{\substack{Q \subset \mathbb{R}^2 \\ Q \text{ is a cube}}} ⨍_Q e^{|f-f_Q|}$$

For a fixed f, β is finite by $\beta \leq e^{2\|f\|_\infty}$, thus β is well defined. That is why we assume f is smooth at first.

Then, notice $|Q_i| \leq \frac{1}{\mu} \int_{Q_i} |f - f_Q|$, and $\sum \int_{Q_i} |f - f_Q| \leq \int_Q |f - f_Q| \leq |Q|$ since $\|f\|_{BMO} = 1$. The inequality shows

$$\begin{aligned} ⨍_Q e^{\alpha|f-f_Q|} &\leq \frac{|Q \backslash \cup Q_i|}{|Q|} e^{\alpha\mu} + \sum_{Q_i} e^{2^n \alpha\mu} \frac{|Q_i|}{|Q|} ⨍_{Q_i} e^{\alpha|f-f_{Q_i}|} \\ \Longrightarrow ⨍_Q e^{\alpha|f-f_Q|} &\leq e^{\alpha\mu} + e^{2^n \alpha\mu} \frac{\sum_{Q_i} \int_{Q_i} |f - f_{Q_i}|}{\mu|Q|} ⨍_{Q_i} e^{\alpha|f-f_{Q_i}|} \\ \Longrightarrow ⨍_Q e^{\alpha|f-f_Q|} &\leq e^{\alpha\mu} + e^{2^n \alpha\mu} \frac{1}{\mu} \beta \end{aligned}$$

Take sup on the left, we have

$$\beta \leq e^{\alpha\mu} + \frac{e^{2^n \alpha\mu}}{\mu} \beta$$

Choose $\mu = 100$, α very small such that $e^{2^n \alpha\mu} < \mu/2$. We have $\beta \leq 2e^{\alpha\mu} < +\infty$. This completes the proof. $\square$

Remark 1. *For the case f is non-smooth, β is not well-defined, i.e. β could be $+\infty$, we only need to mollify f with a sequence of smooth functions f_ε such that $f_\varepsilon \to f$ in L^∞. We have*

$$⨍_Q |f_\varepsilon - (f_\varepsilon)_Q| \leq ⨍ |f - f_Q| + 2\varepsilon$$

Thus if $\|f\|_{BMO} \leq 1$, we could take a seq of f_ε with $\|f_\varepsilon\|_{BMO} \leq 2$, then use the uniform bound of $⨍ e^{\alpha|f_\varepsilon - (f_\varepsilon)_Q|} \leq C$ and Fatou's lemma, we also have

$$⨍_Q e^{\alpha|f-f_Q|} \leq C$$

Lemma 2. *(Caldrón–Zygmund Decomposition)*

For a positive function $f \in L^1(\mathbb{R}^n)$, $\lambda > 0$ is any positive number. We can find a sequence of disjoint cubes Q_i with edges parallel to the coordinate axis, such that

1. $\lambda < \fint_{Q_i} f < 2^n \lambda$ *for any* Q_i
2. $f \leq \lambda$ *in* $\mathbb{R}^n \backslash \cup Q_i$

If $f \in L^1(Q)$ for some bounded cube Q instead of $L^1(\mathbb{R}^n)$, we have the same conclusion for $\lambda > \fint_Q f$.

Proof. Firstly, Choose $r > 0$ large such that $\|f\|_{L^1(\mathbb{R}^n)} < \lambda r^n$, then we have for a cube Q with side length r, we have

$$\fint_Q f \leq \frac{1}{r^n} \int_{\mathbb{R}^n} f < \lambda$$

Decompose $\mathbb{R}^n$ into cubes of radius r, $\mathbb{R}^n = \cup Q_r$ with $Q_r = [k_1 r, (k_1+1)r] \times [k_2 r, (k_2+1)r] \cdots \times [k_n r, (k_n+1)r]$, $k_i \in \mathbb{Z}$. Now we reduce the case into bounded cubes, let's take $Q = [0, r]^n$ for example.

We decompose Q into dyadic sub-cubes: $Q_j^{(1)} = [\frac{k_i}{2} r, \frac{k_i+1}{2} r]^n$, $k_i \in \{0, 1\}$. Consider in $Q_j^{(1)}$, we have

$$\fint_{Q_j^{(1)}} f < 2^n \fint_Q f < 2^n \lambda$$

We keep the cubes of $\{Q_j^{(1)} : \fint_{Q_j^{(1)}} f \geq \lambda\}$, and decompose the rest cubes of $Q_j^{(1)}$ again into dyadic cubes $Q_j^{(2)}$, we still have

$$\fint_{Q_j^{(2)}} f < 2^n \fint_{Q_j^{(1)}} f < 2^n \lambda$$

Repeat the progress, keep $Q_j^{(k)}$ on which the average integral of f is greater than λ, and decompose the rest into smaller cudes. We obtained a sequence of Q_j in Q such that

$$\lambda < \fint_{Q_j} f < 2^n \lambda$$

For a.e. $x \in Q \backslash \cup Q_j$,

$$f(x) = \lim_{\substack{x \in Q_j^{(n)} \\ n \to \infty}} \fint_{Q_j^{(n)}} f \leq \lambda$$

□